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CRITICAL COMBINATIONS OF SHEAR AND DIRECT STRESS FOR

SIMPLY SUPPORTED RECTANGULAR FLAT PLATES

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SUMMARY

The buckling of a simply supported rectangular flat plate
under combinations of shear and direct stress was investigated by
means of an energy method. The critical combinations of stress for
several length-width ratios were determined to an accuracy of about
1 percent by the use of tenth-order determinants in conjunction with
a modified matrix iteration method. Curves were drawn which can be
used to obtain the critical stress combinations for the case of
interaction of shear and longitudinal direct stress and for the case
of interaction of shear and transverse direct stress.

INTRODUCTION

The problem of buckling of plates under the action of more than
one stress has been given considerable attention. For plates
subjected to two stresses the critical combinations are usually given
by means of interaction curves, that is, curves that can be used to
obtain the value of one stress required to produce buckling when a
given value of another stress is also present. The stresses are
ordinarily given nondimensionally in terms of either stress ratios
or stress coefficients.

In reference 1 the interaction curve for infinitely long flat
plates under combined shear and longitudinal direct stress was
shown to be very nearly a parabola as indicated in figure 1 of the
present paper. In this figure R_s is the ratio of shear stress
present to the critical stress in pure shear and R_x is the ratio
of the longitudinal direct stress present to the critical stress
in pure longitudinal compression. (All symbols are defined in
appendix A.) This curve is shown in figure 1 of reference 1 to
apply to plates having edges either simply supported, clamped, or
elastically restrained against rotation. Reference 2 demonstrates

that the parabola also applies, to a high degree of accuracy, to
nearly square plates with simply supported edges. On the other
hand, for infinitely long plates loaded in shear and transverse
direct stress, the interaction curve assumes a different form,
which is shown for simply supported edges in figure 2 taken from
figure 3, reference 3. The form of the interaction curve for shear
and transverse direct stress appears, therefore, to change markedly
as the length-width ratio of the plate increases from 1 to ∞.
The purpose of the present paper is to investigate this transition
and, in addition, to determine whether any appreciable change in
the form of the interaction curve for shear and longitudinal stress
occurs as the length-width ratio of the plate increases from 1
to ∞.

RESULTS AND DISCUSSION

The results discussed herein are based on the theoretical
solution presented in appendix B. The numerical results, computed
by means of the matrix iteration method described in appendix C,
are believed to be accurate to within about 1 percent. Interaction
curves are presented for the buckling of simply supported plates
having length-width ratios in the range of 1 to 4 and subjected to
shear and longitudinal stress and to shear and transverse stress.

Shear and Longitudinal Stress

The interaction curves for shear and longitudinal direct stress
are shown in figure 3 in terms of stress ratios for simply supported
flat rectangular plates having length-width ratios of 1, 2, and 4.
One of the two curves given for each length-width ratio represents
a buckle pattern that is symmetric about the center point of the
plate. The other curve represents a buckle pattern that is anti-
symmetric about the center point of the plate. Those parts of the
two curves that are governing parts (the parts that give lower
values of one stress for a given value of the other stress) are
drawn with solid lines. The purpose of presenting two curves
instead of only the governing parts of either curve is to indicate
where the cusps occur and to show the abrupt change from a symmetric
to an antisymmetric buckle pattern that results in these cusps.

For purposes of comparison, points from the parabola represented
by the following equation for the interaction curve for a length-
width ratio of ∞

$$R_s^2 + R_x = 1 \quad \text{(see reference 1)}$$

are included in each graph in figure 3. The curves for the rectangular plates do not differ visibly from the parabola; this close agreement indicates that for shear and longitudinal direct stress of simply supported rectangular plates the parabolic interaction formula is substantially correct.

Shear and Transverse Stress

The interaction curves in stress-ratio form for shear and transverse direct stress for simply supported rectangular plates having length-width ratios of 2, 3, and 4 are shown in figure 4. The curve for a length-width ratio of 2 is nearly a parabola in the compression range but deviates considerably from the parabola in the tension range. The curves for the higher length-width ratios deviate considerably from a parabola in both compression and tension and incline toward the curve for infinitely long plates. In figure 5 the curves of figure 4, together with curves for plates of length-width ratios of 1 and ∞, are plotted on the same graph. The vertical straight-line part of the interaction curve for a length-width ratio of ∞ corresponds to buckling of the plate as an Euler strip. (See reference 3.)

In figure 6 the transition in the form of the interaction curve as the length-width ratio of the plate changes from 1 to ∞ is shown in terms of buckling stress coefficients. The assumption that a long plate is infinitely long is seen to lead to a conservative estimate of the buckling stresses; if the length-width ratio is greater than 4, only a small error is involved in this assumption.

The combinations of stress coefficients that result in buckling and the corresponding deflection coefficients are given in table 1.

CONCLUSIONS

The results for the critical combinations of direct stress and shear of simply supported rectangular flat plates, computed by an energy method summarized in tables and graphs, show that:

1. For shear and longitudinal direct stress the interaction curve for all length-width ratios investigated is substantially a parabola for which the equation in terms of stress ratios is

$$R_s^2 + R_x = 1$$

where R_s is the ratio of shear stress present to the critical stress in pure shear and R_x is the ratio of longitudinal direct stress present to the critical stress in pure longitudinal compression.

2. For shear and transverse direct stress, the shape of the interaction curve depends on the length-width ratio of the plate. For square plates the interaction curve is very nearly a parabola, the equation for which is given in the preceding paragraph. The interaction curve for plates having a length-width ratio of 2 is nearly parabolic in the compression range but is close to the known interaction curve for infinitely long plates in the tension range. In the range of length-width ratio from 2 to 4 the parabolic equation does not hold even in compression; therefore the curves given in the present paper should be used. At length-width ratios greater than 4, the interaction curve approximates the interaction curve for a length-width ratio of ∞.

Langley Memorial Aeronautical Laboratory
      National Advisory Committee for Aeronautics
            Langley Field, Va., November 8, 1946

APPENDIX A

SYMBOLS

a	length of plate
b	width of plate; $b \leqq a$
t	thickness of plate
w	deflection normal to plate
D	flexural stiffness of plate $\left(\dfrac{Et^3}{12(1 - \mu^2)} \right)$
x	longitudinal coordinate
y	transverse coordinate
B	symmetrical matrix
E	Young's modulus for material
a_{mn}	deflection-function coefficient
b_{mn}	element of matrix B
L_{mn}	diagonal terms in stability determinant
x_m	mth unknown in set $x_1, x_2, \cdots x_N$
i,j,m,n,p,q,N	integers
C_i	arbitrary coefficient
σ_x	longitudinal compressive stress
σ_y	transverse compressive stress
τ	shear stress
R_x	longitudinal direct-stress ratio
R_y	transverse direct-stress ratio

R_s shear-stress ratio

k_x longitudinal compressive-stress coefficient

k_y transverse compressive-stress coefficient

k_s shear-stress coefficient

β length-width ratio (a/b)

μ Poisson's ratio for material

λ characteristic value of matrix B

λ_i ith characteristic value of matrix B

ξ column matrix with elements $x_1, x_2, \ldots x_N$

ξ_i modal column associated with ith characteristic value

$\xi^{(n)}$ nth approximation to ξ.

$\bar{\xi}^{(n)}$

        column matrix constructed so that each element is the mean of

            the corresponding elements of $\xi^{(n)}$ and $\xi^{(n-1)}$

cr   critical (used as subscript)

APPENDIX B

THEORETICAL SOLUTION

Method of Analysis

The problem of the stability of finite rectangular flat plates under combined shear and direct stress is solved by use of the Rayleigh-Ritz method. (See reference 4.) The deflection function is expressed exactly by means of a two-dimensional infinite Fourier series. The application of the Rayleigh-Ritz method results in an infinite set of homogeneous linear equations in the infinite number of unknown Fourier coefficients. Solutions of these equations that give Fourier coefficients not all equal to zero exist only for those combinations of shear and direct stress for which the buckled plate is in neutral equilibrium.

The algebraic equations obtained herein are equivalent to the equations used in reference 2. In order to obtain accurate results for relatively long plates, however, a method of solution of these equations different from the method of reference 2 was employed in the present paper. In reference 2, two sixth-order determinants (one determinant corresponding to a symmetrical and the other determinant to an antisymmetrical buckle pattern) were expanded and the resulting polynomials were solved for the critical stress combinations. Each solution involved the use of only six terms in the Fourier expansion of the deflection function. In the present paper 10 terms were used in the expansion of the deflection function. The corresponding set of 10 simultaneous equations was solved by the matrix interation method (references 4 and 5) modified in the manner described in appendix C. This method has the advantages that, whereas a very accurate solution requires a great deal of labor, a good approximate solution can be obtained without much effort, and the Fourier coefficients as well as the buckling loads can be evaluated. A knowledge of the Fourier coefficients for a given loading condition is used to determine whether the best choice of equations was made and also to assist in the choice of the 10 most important equations for a similar loading condition.

Accuracy of Results

Difficultly is usually encountered in evaluating the discrepancy between the buckling load determined by an exact solution of the 10 most important equations and the true value of the buckling load.

On the assumption that the use of the 10 most important simultaneous
equations of the infinite set to find the critical shear stress for
a given compression stress would in all cases result in an error of
not more than a few tenths of 1 percent, the iteration (see appendix C)
was carried out until the error in the solution of the equations used
was estimated to be less than 0.5 percent. The shear-stress
coefficients presented in table 1 of the present paper therefore
are believed to be in error by not more than 1 percent.

Solution

The critical stresses are determined on the basis of the principle
that during buckling the elastic-strain energy stored in a structure
is equal to the work done by the applied loads. For the case of a
rectangular flat plate under loads applied in the plane of the plate,
this equality becomes, when the coordinate system is that of figure 7,

$$\frac{D}{2}\int_0^b\int_0^a\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dx\,dy$$

$$=\frac{t}{2}\int_0^b\int_0^a\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2+\sigma_y\left(\frac{\partial w}{\partial y}\right)^2+2\tau\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy \qquad (B1)$$

(equivalent to equation 210, reference 6)

Equation (B1) can be rewritten in terms of nondimensional stress
coefficients as follows:

$$\int_0^b\int_0^a\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right.$$

$$\left.-k_x\left(\frac{\pi}{b}\right)^2\left(\frac{\partial w}{\partial x}\right)^2-k_y\left(\frac{\pi}{b}\right)^2\left(\frac{\partial w}{\partial y}\right)^2-2k_s\left(\frac{\pi}{b}\right)^2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}dx\,dy=0 \qquad (B2)$$

where

$$k_x = \sigma_x \frac{b^2 t}{\pi^2 D}$$

$$k_y = \sigma_y \frac{b^2 t}{\pi^2 D}$$

$$k_s = \tau \frac{b^2 t}{\pi^2 D}$$

The procedure used in solving equation (B2) involves substituting for w a function of x and y that satisfies the edge conditions and can be adjusted so as to closely approximate the buckling configuration. For any case in which the value of w is 0 at all the edges, the term with the coefficient $-2(1 - \mu)$ can be shown to vanish (reference 7). A series of terms with arbitrary coefficients is used to represent w, and the coefficients are determined by the Rayleigh-Ritz method. A general form for the deflection w is

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (B3)$$

Equation (B2) is solved for the case of buckling under shear and transverse stress and for the case of buckling under shear and longitudinal stress.

Shear and transverse direct stress.- If the value of k_x is set equal to 0 and the expression for w is substituted in equation (B2), the following equation is obtained:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a^2_{mn} \frac{ab}{4} \left\{ \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2 - k_y \left(\frac{\pi}{b}\right)^2 \left(\frac{n\pi}{b}\right)^2 \right\}$$

$$-2k_s \left(\frac{\pi}{b}\right)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{mn} a_{pq} \frac{4ab}{\pi^2} \left(\frac{m\pi}{a}\right) \left(\frac{q\pi}{b}\right) \frac{pn}{(p^2 - m^2)(n^2 - q^2)} = 0$$

$$(B4)$$

where $m \pm p$ and $n \pm q$ are odd numbers.

The coefficient a_{mn} must be chosen to make the value of k_s a minimum. This procedure results in the set of homogeneous linear equations represented by

$$a_{mn} \left[\left(m^2 + n^2\beta^2\right)^2 - k_y n^2\beta^4 \right] + \frac{32k_s\beta^3}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} = 0$$

$$(B5)$$

where

$$\beta = \frac{a}{b}$$

$$m = 1, 2, 3, \ldots$$

$$n = 1, 2, 3, \ldots$$

and $m \pm p$ and $n \pm q$ are odd numbers.

This set of equations may be divided into two groups which are independent of each other, one group in which $m \pm n$ is odd (antisymmetric buckling), and one group in which $m \pm n$ is even (symmetric buckling). Ten equations in 10 unknowns were solved for each group by the iteration method explained in appendix C.

A representative determinant in terms of the coefficients for the group of equations in which $m \pm n$ is even is

	a_{11}	a_{13}	a_{22}	a_{31}	a_{15}	a_{24}	a_{33}	a_{42}	a_{51}	a_{44}
m=1, n=1	L_{11}	0	$\frac{4}{9}$	0	0	$\frac{8}{45}$	0	$\frac{8}{45}$	0	$\frac{16}{225}$
m=1, n=3	0	L_{13}	$-\frac{4}{5}$	0	0	$\frac{8}{7}$	0	$-\frac{8}{25}$	0	$\frac{16}{35}$
m=2, n=2	$\frac{4}{9}$	$-\frac{4}{5}$	L_{22}	$\frac{4}{5}$	$-\frac{20}{63}$	0	$\frac{36}{25}$	0	$-\frac{20}{63}$	0
m=3, n=1	0	0	$-\frac{4}{5}$	L_{31}	0	$-\frac{8}{25}$	0	$\frac{8}{7}$	0	$\frac{16}{35}$
m=1, n=5	0	0	$-\frac{20}{63}$	0	L_{15}	$-\frac{40}{27}$	0	$-\frac{8}{63}$	0	$-\frac{16}{27}$
m=2, n=4	$\frac{8}{45}$	$\frac{8}{7}$	0	$-\frac{8}{25}$	$-\frac{40}{27}$	L_{24}	$-\frac{72}{35}$	0	$-\frac{8}{63}$	0
m=3, n=3	0	0	$\frac{36}{25}$	0	0	$-\frac{72}{35}$	L_{33}	$-\frac{72}{35}$	0	$\frac{144}{49}$
m=4, n=2	$\frac{8}{45}$	$-\frac{8}{25}$	0	$\frac{8}{7}$	$-\frac{8}{63}$	0	$-\frac{72}{35}$	L_{42}	$-\frac{40}{27}$	0
m=5, n=1	0	0	$-\frac{20}{63}$	0	0	$-\frac{8}{63}$	0	$-\frac{40}{27}$	L_{51}	0
m=4, n=4	$\frac{16}{225}$	$\frac{16}{35}$	0	$\frac{16}{35}$	$-\frac{16}{27}$	0	$\frac{144}{49}$	0	0	L_{44}

(B6)

where

$$L_{mn} = \frac{\pi^2}{32 k_s \beta^3}\left[\left(m^2 + n^2\beta^2\right)^2 - k_y n^2 \beta^4\right]$$

A typical determinant in terms of the coefficients for the group of equations in which $m \pm n$ is odd is

	a_{12}	a_{21}	a_{14}	a_{23}	a_{32}	a_{41}	a_{25}	a_{34}	a_{43}	a_{52}
m=1, n=2	L_{12}	$-\dfrac{4}{9}$	0	$\dfrac{4}{5}$	0	$-\dfrac{8}{45}$	$\dfrac{20}{63}$	0	$\dfrac{8}{25}$	0
m=2, n=1	$-\dfrac{4}{9}$	L_{21}	$-\dfrac{8}{45}$	0	$\dfrac{4}{5}$	0	0	$\dfrac{8}{25}$	0	$\dfrac{20}{63}$
m=1, n=4	0	$-\dfrac{8}{45}$	L_{14}	$-\dfrac{8}{7}$	0	$-\dfrac{16}{225}$	$\dfrac{40}{21}$	0	$-\dfrac{16}{35}$	0
m=2, n=3	$\dfrac{4}{5}$	0	$-\dfrac{8}{7}$	L_{23}	$-\dfrac{36}{25}$	0	0	$\dfrac{72}{35}$	0	$-\dfrac{4}{7}$
m=3, n=2	0	$\dfrac{4}{5}$	0	$-\dfrac{36}{25}$	L_{32}	$-\dfrac{8}{7}$	$-\dfrac{4}{7}$	0	$\dfrac{72}{35}$	0
m=4, n=1	$-\dfrac{8}{45}$	0	$-\dfrac{16}{225}$	0	$-\dfrac{8}{7}$	L_{41}	0	$-\dfrac{16}{35}$	0	$\dfrac{40}{27}$
m=2, n=5	$\dfrac{20}{63}$	0	$\dfrac{40}{21}$	0	$-\dfrac{4}{7}$	0	L_{25}	$-\dfrac{8}{3}$	0	$\dfrac{100}{441}$
m=3, n=4	0	$\dfrac{8}{25}$	0	$\dfrac{72}{35}$	0	$-\dfrac{16}{35}$	$-\dfrac{8}{3}$	L_{34}	$-\dfrac{144}{49}$	0
m=4, n=3	$\dfrac{8}{25}$	0	$-\dfrac{16}{35}$	0	$\dfrac{72}{35}$	0	0	$-\dfrac{144}{49}$	L_{43}	$-\dfrac{8}{3}$
m=5, n=2	0	$\dfrac{20}{63}$	0	$-\dfrac{4}{7}$	0	$\dfrac{40}{27}$	$\dfrac{100}{441}$	0	$-\dfrac{8}{3}$	L_{52}

(B7)

In general, the method is to choose numerical values of β and k_y, set each determinant equal to 0, and solve for the lowest value of k_s. The lower of the two values of k_s found from the two determinants establishes the critical shear stress for a panel with length-width ratio β under the transverse stress given by k_y.

Shear and longitudinal direct stress.- If k_y is set equal
to 0 in equation (B2) and the same procedure as the procedure
for the combination of shear and transverse stress is carried out,
the set of homogeneous linear equations given by the Rayleigh-Ritz
method is represented by

$$a_{mn}\left[\left(m^2 + n^2\beta^2\right)^2 - k_x m^2\beta^2\right] + \frac{32k_s\beta^3}{\pi^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} a_{pq}\frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} = 0$$

(B8)

where

$m = 1, 2, 3, \ldots$

$n = 1, 2, 3, \ldots$

and $m \pm p$ and $m \pm q$ are odd numbers.

The determinants set up from these equations are the same as the
determinants (B6) and (B7) for symmetric and antisymmetric buckling,
respectively, except that the diagonal terms are

$$I_{mn} = \frac{\pi^2}{32k_s\beta^3}\left[\left(m^2 + n^2\beta^2\right)^2 - k_x m^2\beta^2\right]$$

In general, the method is to choose numerical values of β
and k_x, set each determinant equal to 0, and solve for the
lowest value of k_s. As in the determination of the shear and
transverse direct stress, the lowest value of k_s establishes the
critical shear stress for a panel with length-width ratio β under
the longitudinal stress given by k_x.

APPENDIX C

MATRIX ITERATION METHOD

If the matrix iteration method is used in attempting to solve the set of simultaneous linear algebraic equations associated with the buckling of flat plates where shear is present, the conventional iteration process as described in reference 4 does not converge. The reason for this nonconvergence, the modification in the iteration process used in the present paper to obtain convergence, and the method of choosing the best finite set of algebraic equations to use are described in the following paragraphs.

Conventional Iteration Process

The matrix iteration method is described in the present paper in terms of the following simplified symmetrical equation:

$$\sum_{n=1}^{\infty} b_{mn}x_n = \lambda x_m \qquad (C1)$$

where

$$m = 1, 2, 3, \ldots$$

and

$$b_{mn} = b_{nm}$$

This set of equations is equivalent to the set of equations (B5) from which the critical shear stress is determined if

$$x_m = a_{ij} \sqrt{(i^2 + j^2\beta^2)^2 - k_y i^2\beta^4} \qquad (C2)$$

(where m is a different integer for each different combination of i and j),

$$b_{mn} = \frac{32\beta^3}{\pi^2} \frac{ijpq}{(i^2 - p^2)(j^2 - q^2)\left[(i^2 + j^2\beta^2)^2 - k_y j^2\beta^4\right]^{1/2}(pq)^{1/2}}$$

$$(C3)$$

(where n bears the same relation to p, q as m bears to i, j),
and

$$\lambda = \frac{1}{k_s} \tag{C4}$$

Equations (C1) can similarly be shown to be a generalization of
equations (B8).

The problem is to find the highest value of λ which permits
nonvanishing solutions to be found for equation (C1). In order to
avoid the use of too many suffixes, matrix notation is used, so that
equation (C1) becomes

$$B\xi = \lambda\xi \tag{C5}$$

where B is a square matrix and ξ is a column vector.

The matrix iteration method (reference 5) consists in taking
some arbitrarily assumed values for the set of values for ξ
and in calculating the left-hand side of equation (C5) in order to
obtain an improved set of values for ξ. (Division by λ is
unnecessary because the solution of homogeneous equations can be
determined only to within an arbitrary multiplicative constant.)
These new values are reinserted in the left-hand side of equation (C5)
to obtain further improvement in the values, and the process is
continued until the ratios of the components of ξ are not
appreciably changed by further iterations. The value of λ is then
given as the ratio of the last value found for ξ to the preceding
value.

The basis for this method can be seen from the following
discussion. According to matrix theory, if the matrix B is
of Nth order, there are N values of λ satisfying equation (C5).
Let these values be called λ_1, λ_2, ..., λ_N with the order so
arranged that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_N|$. For each value of λ_1 there
is a corresponding solution for x, which may be called ξ_1. Then

$$B\xi_1 = \lambda_1\xi_1 \tag{C6}$$

Any initial assumed solution $\xi^{(0)}$ can be expressed in terms of the N true solutions $\xi_1, \xi_2, \ldots, \xi_N$ as follows:

$$\xi^{(0)} = \sum_{i=1}^{N} c_i \xi_i \tag{C7}$$

where the values of the coefficients c_i are in general not known. The result of one iteration, obtained by using equations (C6) and (C7), is seen to be

$$\xi^{(1)} = B\xi^{(0)}$$

$$= \sum_{i=1}^{N} c_i B \xi_i$$

$$= \sum_{i=1}^{N} c_i \lambda_i \xi_i$$

The result of n iterations is

$$\xi^{(n)} = \sum_{i=1}^{N} c_i \lambda_i^{n} \xi_i$$

The next step is to factor out the nth power of λ_1, that is, the largest value of λ:

$$\xi^{(n)} = \lambda_1^{n} \left[c_1 \xi_1 + \sum_{i=2}^{N} c_i \left(\frac{\lambda_i}{\lambda_1} \right)^{n} \xi_i \right] \tag{C8}$$

Equation (C8) shows that as n increases $\underline{\xi}^{(n)}$ becomes more and more nearly a constant times $\underline{\xi}_1$, the rate of convergence depending on the smallness of the ratio $\left|\frac{\lambda_i}{\lambda_1}\right|$ $(i = 2, 3, \ldots, N)$. The largest value of λ can then be found as

$$\lambda_1 = \frac{\xi^{(n)}}{\xi^{(n-1)}}$$

Modification for Shear Buckling Problems

In plate buckling problems in which shear is present, the critical shear stresses occur in pairs which are equal in magnitude but opposite in sign. For such a problem $\lambda_2 = -\lambda_1$, and equation (C8) may be written

$$\xi^{(n)} = \lambda_1^n \left[c_1 \underline{\xi}_1 + (-1)^n c_2 \underline{\xi}_2 + \sum_{i=3}^{N} c_i \left(\frac{\lambda_i}{\lambda_1}\right)^n \underline{\xi}_i \right] \tag{C9}$$

Equation (C9) implies that unless c_2 is by chance equal to 0, no amount of iteration will result in convergence.

A simple expedient can be used, however, to produce convergence. Since equation (C9) shows that as n increases $\underline{\xi}^{(n)}$ oscillates about the true solution (constant times $\underline{\xi}_1$), an improved approximation can be obtained by constructing $\overline{\xi}^{(n)}$ such that each component is some kind of mean of the corresponding components of $\underline{\xi}^{(n)}$ and $\underline{\xi}^{(n-1)}$. Immediate convergence results from the use of the geometric mean (but not from the use of the arithmetic mean) after one iteration if only two equations of the set for a rectangular plate in shear are used. In the present paper, therefore, in which 10 equations were used, the geometric mean was employed except when Fourier coefficients of two successive iterations had opposite signs, in which case the arithmetic mean was used.

The procedure adopted for obtaining the shear buckling stress was to use the matrix iteration method modified by taking the geometric mean after every two iterations as described in the preceding paragraph. In addition, by use of a method suggested in

reference 8, an improved value for λ_1 was obtained after the
completion of n iterations by means of the equation

$$\lambda_1 = \frac{\left[\xi^{(n)}\right]^2}{\xi^{(n)} \xi^{(n-1)}}$$

Choice of Equations

An exact solution of the critical shear stress for rectangular
plates involves the use of an infinite set of equations in an
infinite number of unknowns. Since attention must be confined to
a finite number of equations – say, N – the ability to choose the
best N equations for the purpose is very desirable.

A very useful (although not rigorously correct) guide to
the best choice of the equations to be used may be obtained from a
consideration of the accuracy of representation of the buckle
deformation. The use of N equations in N unknowns implies that
the deflection surface is being described in terms of N Fourier
components, with the other components assumed equal to 0.

The matrix iteration method yields the Fourier coefficients as
well as the critical stress coefficient. The values found for these
Fourier coefficients (where N was taken to be 10) were substituted
in the following form:

m ± n   even

			a_{17}
		a_{15}	a_{26}
	a_{13}	a_{24}	a_{35}
a_{11}	a_{22}	a_{33}	a_{44}
	a_{31}	a_{42}	a_{53}
	a_{51}	a_{62}	
		a_{71}	

m ± n   odd

			a_{18}
		a_{16}	a_{27}
	a_{14}	a_{25}	a_{36}
a_{12}	a_{23}	a_{34}	a_{45}
a_{21}	a_{32}	a_{43}	a_{54}
a_{41}	a_{52}	a_{63}	
	a_{61}	a_{72}	
		a_{81}	

As a result of this substitution, values were inserted in the
10 squares corresponding to coefficients assumed not equal to 0,
whereas no values were substituted for the remaining squares.

The absolute values of the coefficients were observed to
decrease in magnitude in a rather uniform manner as the distance
from the largest coefficient increased. If a space in which no
value was substituted occurs in one of the foregoing forms in a
region where the neighboring computed coefficients are not small,
appreciable error is usually incurred by the neglect of that term.
In such cases, the buckling stress was recomputed with the coefficient
in that space included and the smallest coefficient dropped from
consideration.

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TABLE 1

STRESS COEFFICIENTS AND DEFLECTION-FUNCTION COEFFICIENTS

			Shear and longitudinal direct stress									
			$\frac{a}{b} = 1$									
	Stress coefficients k_s		Deflection-function coefficients a_{ij} ($i+j$ even)									
k_x	$i+j$ Even	$i+j$ Odd	a_{11}	a_{13}	a_{22}	a_{31}	a_{15}	a_{24}	a_{33}	a_{42}	a_{51}	a_{35}
4	0	7.05	1	0	0	0	0	0	0	0	0	0
3	4.68	8.42	1	-.023	.143	-.017	-.001	.006	.009	.006	-.001	.001
2	6.62	9.61	1	-.041	.205	-.035	-.003	.007	.019	.006	-.002	.003
1	8.11		1	-.057	.253	-.052	-.004	.006	.028	.005	-.003	.004
0	9.35	11.63	1	-.070	.293	-.071	-.005	.005	.038	.003	-.004	.005
-2	11.56		1	-.091	.360	-.107	-.006	.002	.057	-.003	-.006	.007
-4	13.46	14.76	1	-.107	.417	-.145	-.007·	-.002	.076	-.011	-.007	.008

			$\frac{a}{b} = 2$									
			$i+j$ odd									
			a_{12}	a_{21}	a_{14}	a_{23}	a_{32}	a_{41}	a_{25}	a_{34}	a_{43}	a_{52}
4	2.89	0	0	1	0	0	0	0	0	0	0	0
3	4.19	3.29	-.137	1	-.003	-.018	.149	-.052	-.001	.005	.007	.014
2	5.12	4.66	-.195	1	-.004	-.035	.214	-.085	-.002	.006	.015	.014
1	5.93	5.71	-.239	1	-.004	-.053	.265	-.109	-.003	.006	.024	.011
			$i+j$ even									
			a_{11}	a_{13}	a_{22}	a_{31}	a_{15}	a_{24}	a_{33}	a_{42}	a_{51}	a_{35}
0	6.59	6.61	1	.032	.343	-.325	-.002	.005	.048	-.042	-.010	.004
-2	7.89		1	.042	.377	-.288	-.002	.005	.058	-.036	-.014	.006
-4	9.04	9.48	1	.052	.407	-.269	-.003	.003	.069	-.034	-.016	.008

			$\frac{a}{b} = 4$									
			$i+j$ odd									
			a_{12}	a_{21}	a_{23}	a_{32}	a_{41}	a_{43}	a_{52}	a_{61}	a_{72}	a_{81}
4	1.37	0	0	0	0	0	1	0	0	0	0	0
3	3.06	2.94	-.010	-.224	.004	-.165	1	-.019	.148	-.056	.025	-.029
			$i+j$ even									
			a_{11}	a_{22}	a_{31}	a_{33}	a_{42}	a_{51}	a_{53}	a_{62}	a_{71}	a_{91}
2	4.10	4.12	-.069	.142	1	-.026	-.252	-.334	.019	.013	-.028	-.008
1		5.00										
0	5.67	5.77	-.153	.209	1	-.048	-.324	-.300	.032	.012	-.032	-.010
-4	7.98	8.00	-.315	.316	1	-.090	-.422	-.292	.057	.025	-.026	-.010

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TABLE 1

STRESS COEFFICIENTS AND DEFLECTION-FUNCTION COFFICIENTS - Concluded

Shear and transverse direct stress												

$\frac{a}{b} = 2$

Stress coefficients k_s			Deflection-function coefficients a_{ij}									
			$i + j$ even									
k_y	$1+j$ Even	$1+j$ Odd	a_{11}	a_{13}	a_{22}	a_{31}	a_{15}	a_{24}	a_{33}	a_{42}	a_{51}	a_{35}
1.56	0	4.70	1	0	0	0	0	0	0	0	0	0
1.45	1.93		1	-.002	.076	-.018	.000	.002	.003	.009	-.001	.000
1.3	2.94		1	-.006	.119	-.043	.000	.003	.006	.010	-.005	.001
1.0	4.18		1	-.013	.182	-.102	-.001	.004	.014	.005	-.009	.001
.5	5.65		1	-.023	.264	-.201	-.001	.005	.029	-.012	-.013	.002
0	6.59	6.61	1	-.032	.343	-.324	-.002	.006	.047	-.043	-.010	.004

			$i + j$ odd									
			a_{12}	a_{21}	a_{23}	a_{32}	a_{41}	a_{14}	a_{34}	a_{43}	a_{25}	a_{52}
-.8	7.76	7.52	-.270	1	-.078	.337	-.155	-.005	.005	.040	-.005	.004
-1.5	8.56	8.31	-.264	1	-.085	.361	-.176	-.005	.004	.045	-.006	.001

$\frac{a}{b} = 3$

			$i + j$ even									
			a_{11}	a_{13}	a_{22}	a_{31}	a_{15}	a_{24}	a_{33}	a_{42}	a_{51}	a_{35}
1.23	0	3.59	1	0	0	0	0	0	0	0	0	0
1.1	2.69		1	-.003	.098	-.072	.000	.002	.004	.009	-.010	.000
.9	4.01		1	-.008	.179	-.234	.000	.004	.014	-.017	-.009	.001

			$i + j$ odd									
			a_{12}	a_{21}	a_{23}	a_{32}	a_{41}	a_{14}	a_{34}	a_{43}	a_{25}	a_{52}
.5	5.21	5.04	.153	1	-.032	-.279	-.237	.003	-.006	.027	-.002	.012
0	6.04	5.89	.154	1	-.041	-.337	-.333	.003	-.006	.039	-.002	.035
-.5	6.63	6.60	.147	1	-.048	-.396	-.455	.003	-.007	.054	-.003	.069
-1.2	7.50	7.47	.131	1	-.057	-.487	-.654	.003	-.007	.081	-.004	.131

$\frac{a}{b} = 4$

			$i + j$ even									
			a_{11}	a_{13}	a_{22}	a_{31}	a_{24}	a_{33}	a_{42}	a_{51}	a_{62}	a_{71}
1.13	0		1	0	0	0	0	0	0	0	0	0
1.05	2.61		1	-.002	-.007	-.068	-.002	.002	-.013	-.014	-.004	-.004

			a_{11}	a_{22}	a_{31}	a_{33}	a_{42}	a_{51}	a_{62}	a_{71}	a_{53}	a_{91}
0	5.67	5.77	-.153	.209	1	-.048	.324	-.300	-.012	-.032	.032	-.010
-.5	6.37		-.088	-.198	1	-.055	.370	-.382	-.033	-.034	.042	-.012
-1.1	7.07	7.11	-.027	-.176	1	-.063	.448	-.578	-.095	-.007	.066	-.010

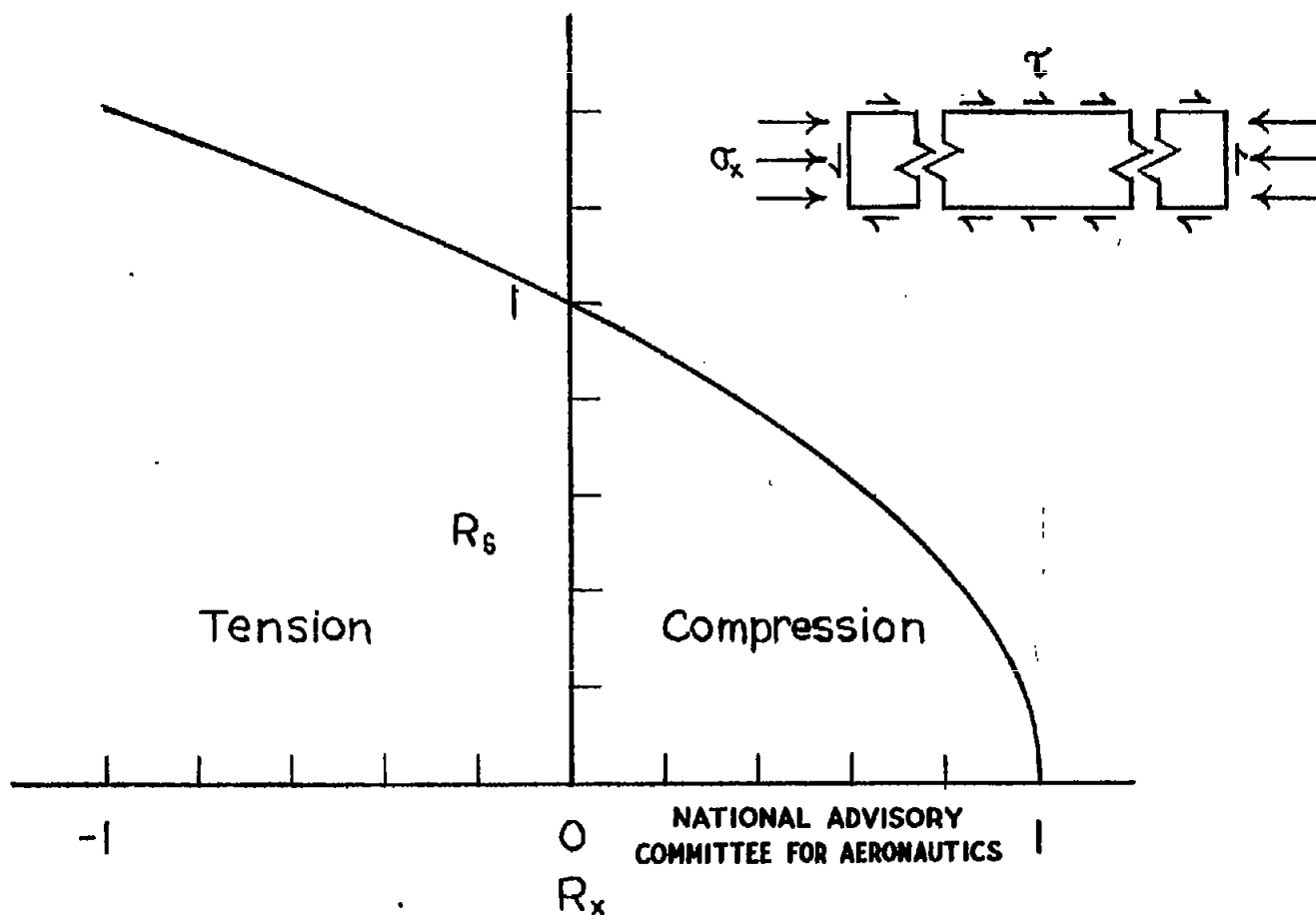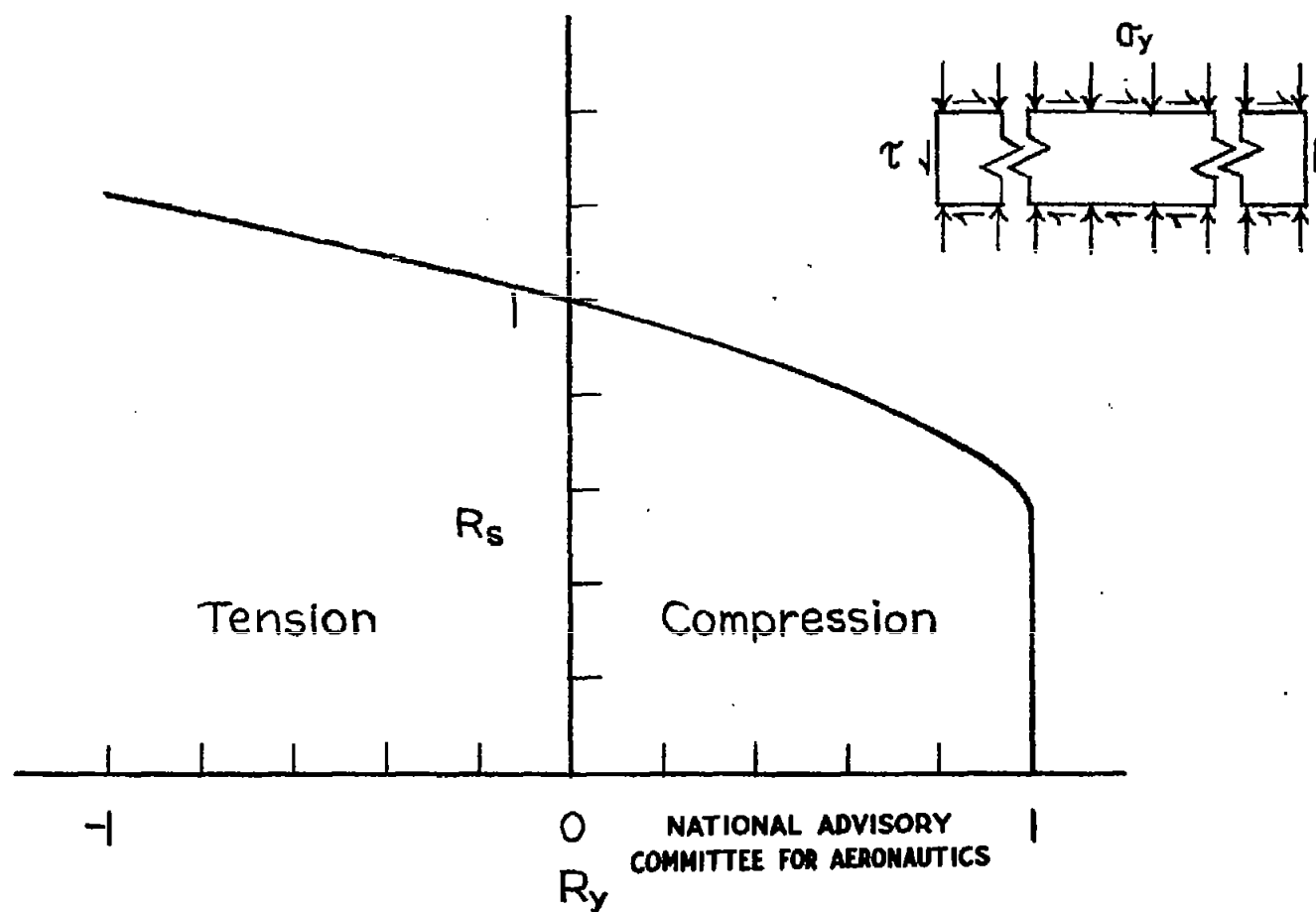|			$i + j$ odd									
			a_{12}	a_{21}	a_{23}	a_{32}	a_{41}	a_{34}	a_{43}	a_{52}	a_{81}	a_{72}
.9	3.99	3.89	.099	1	-.015	-.208	-.243	-.004	.014	.009	-.007	-.005
.5	4.93	4.89	.102	1	-.023	-.299	-.462	-.006	.030	.057	-.005	.005

Figure 1.- Interaction curve for an infinitely long flat plate under combined shear and longitudinal direct stress in terms of stress ratios R_S and R_X taken from figure 1 of reference 1.

$$R_S = \frac{\tau}{\tau_{cr}} \quad ; \quad R_X = \frac{\sigma_X}{\sigma_{X_{cr}}} \quad .$$

Fig. 1

Fig. 2

Figure 2.- Interaction curve for an infinitely long flat plate under combined shear and transverse direct stress in terms of stress ratios R_s and R_y from figure 3 of reference 3.

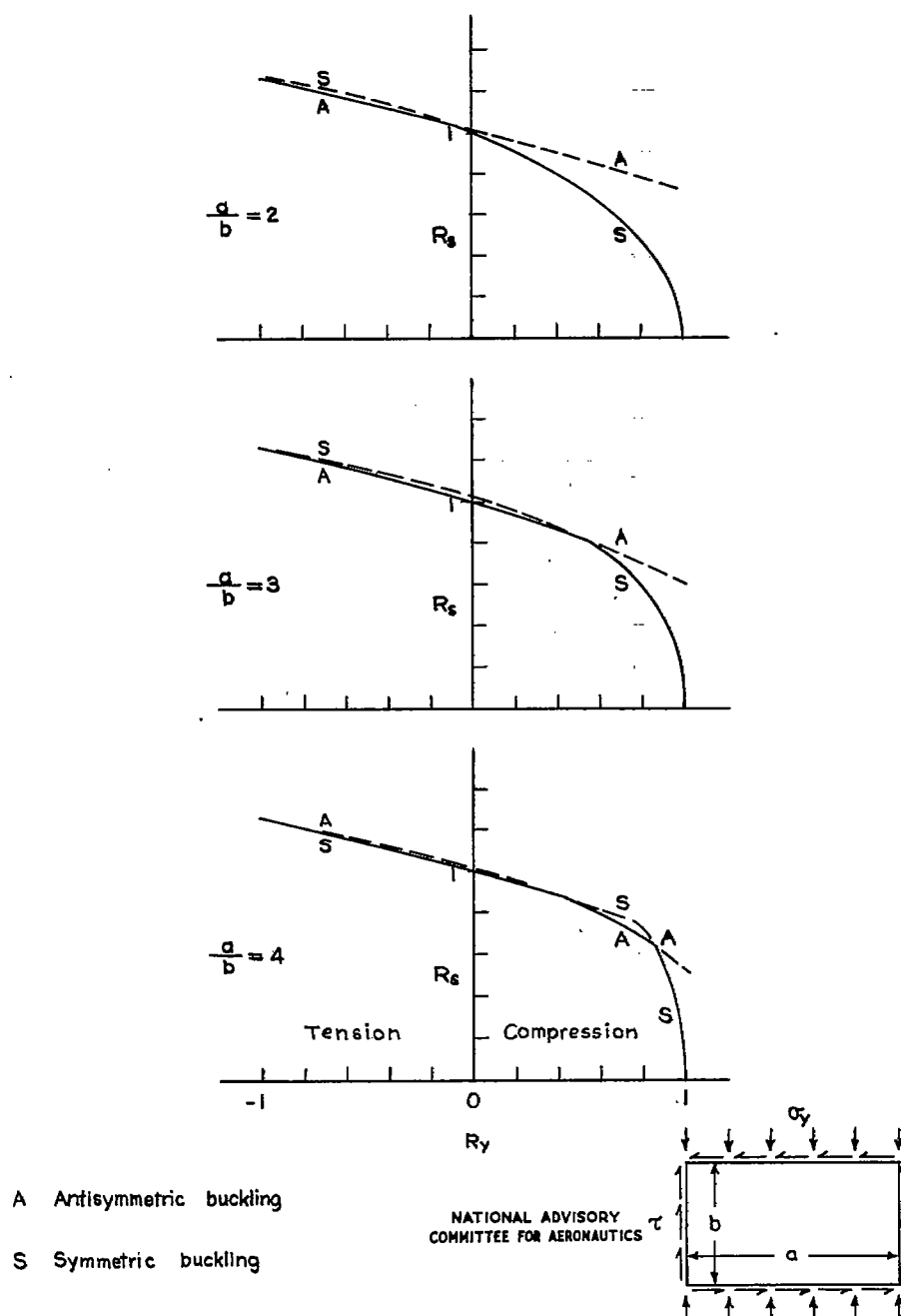$$R_s = \frac{\tau}{\tau_{cr}} \quad ; \quad R_y = \frac{\sigma_y}{\sigma_{y_{cr}}} \; .$$
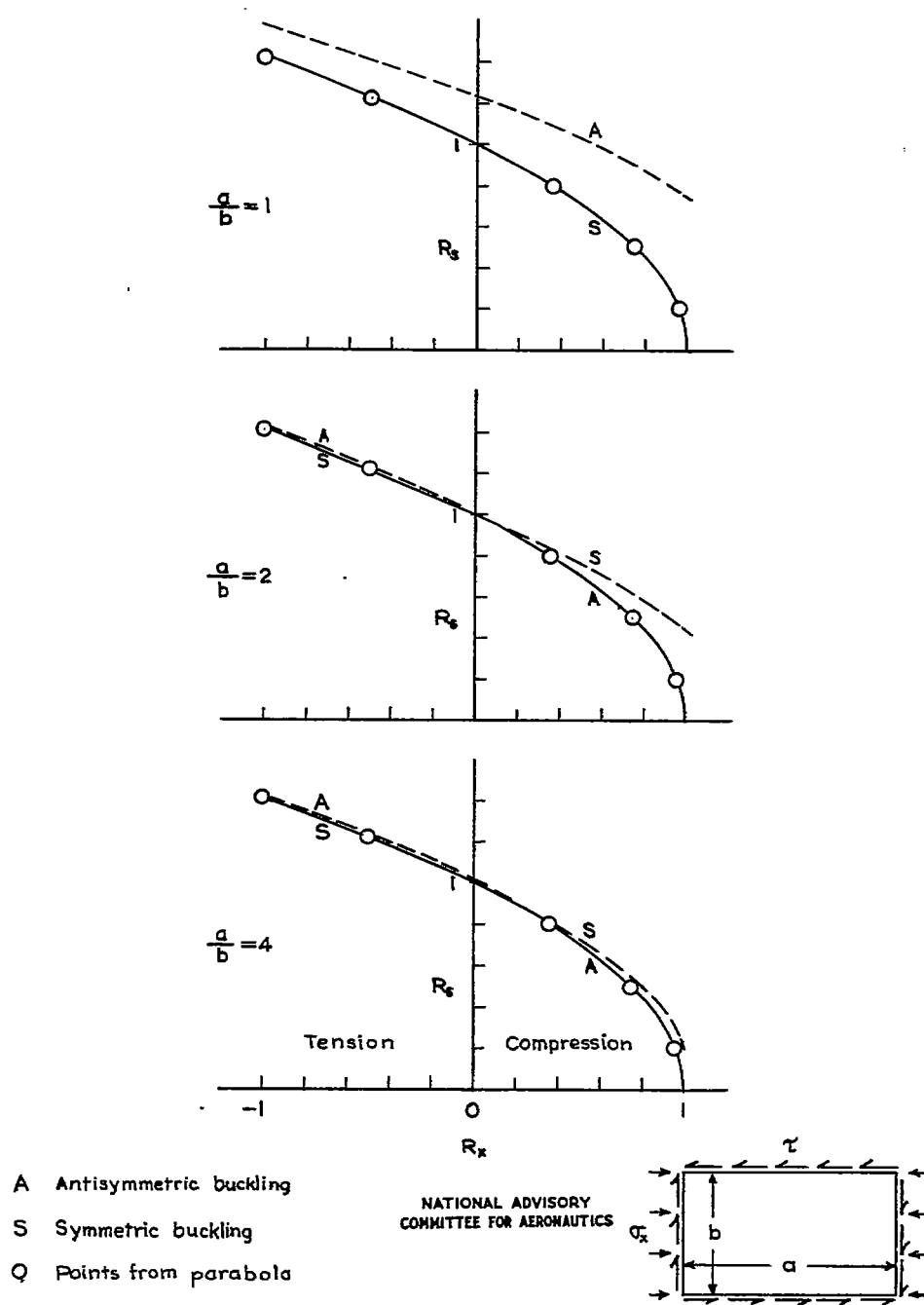
Figure 3.- Interaction curves in terms of stress ratios R_S and R_X for shear and longitudinal direct stress of simply supported rectangular flat plates having length-width ratios of 1, 2, and 4 and comparison with points from the curve representing the parabolic interaction formula $R_S^2 + R_X = 1$. $R_S = \dfrac{\tau}{\tau_{cr}}$; $R_X = \dfrac{\sigma_X}{\sigma_{X_{cr}}}$.

Fig. 4   NACA TN No. 1223

Figure 4.- Interaction curves in terms of stress ratios R_s and R_y for shear and transverse direct stress of simply supported rectangular flat plates having length-width ratios of 2, 3, and 4.

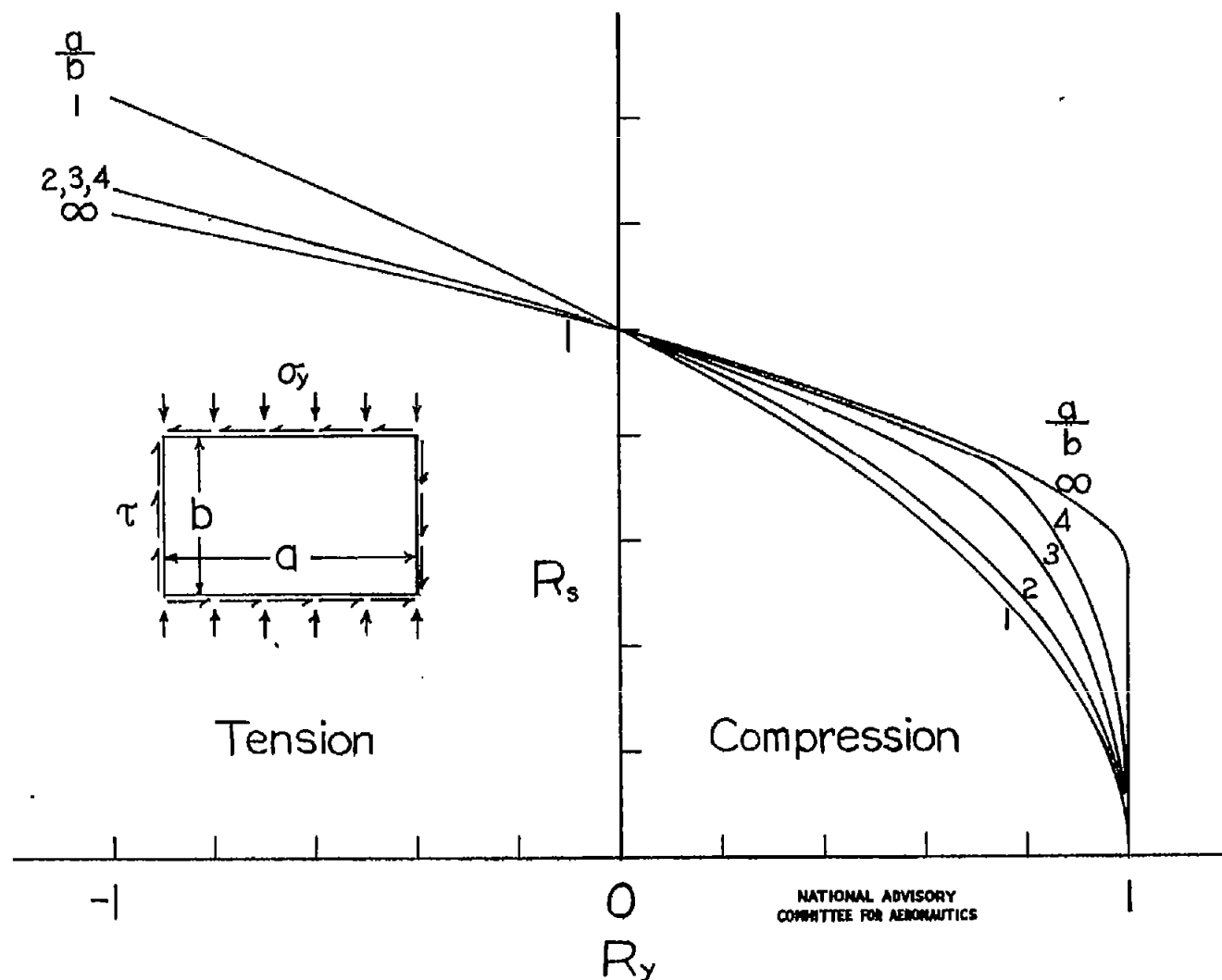$$R_s = \frac{\tau}{\tau_{cr}} \quad ; \quad R_y = \frac{\sigma_y}{\sigma_{y_{cr}}} \; .$$
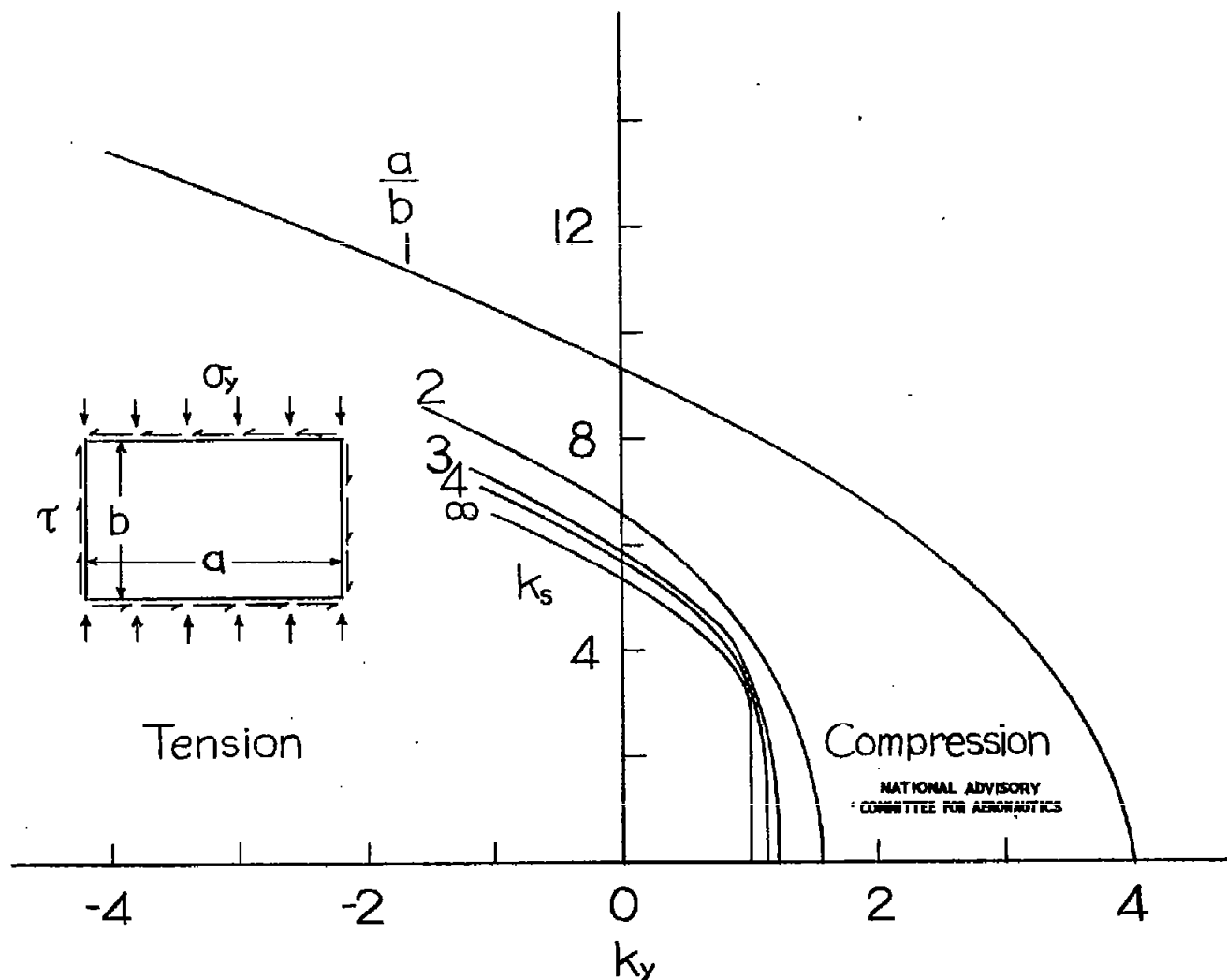
Fig. 5

Figure 5.- Transition in the form of interaction curve for shear and transverse direct stress for a simply supported rectangular flat plate as the length-width ratio changes from 1 to ∞ in terms of

$$R_s \text{ and } R_y. \quad R_s = \frac{\tau}{\tau_{cr}} \quad ; \quad R_y = \frac{\sigma_y}{\sigma_{y_{cr}}} \quad .$$

Fig. 6

Figure 6.- Transition in the form of interaction curve for shear and transverse direct stress for a simply supported rectangular flat plate as the length-width ratio changes from 1 to ∞ in terms of stress coefficients k_s and k_y. $k_s = \dfrac{\tau t b^2}{\pi^2 D}$; $k_y = \dfrac{\sigma_y t b^2}{\pi^2 D}$.
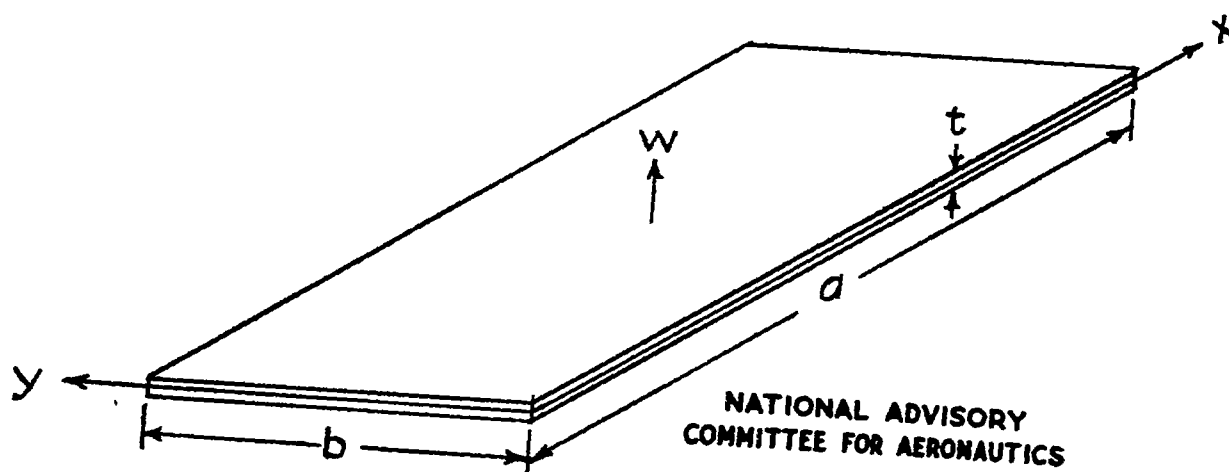
Figure 7.- Coordinate system for a rectangular flat plate.